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Space-charge effects as determined by initial-field conditions

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SPACE-CHARGE EFFECTS
AS DETERMINED BY INITIAL-FIELD CONDITIONS

BY

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A

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ABSTRACT

A method for calculating the current density passing between two parallel plane electrodes for different values of positive charges as a function of " field at the emitter to field for no charge ratio " is presented.

The problem of the potential distribution between the electrodes has been solved for one particular case in which the positive charges are assumed to be proportional to the field at the emitter. The expressions for the electric field, the velocity of charges, and the charge density as a function of the potential are also presented and plotted as a function of the distance.

The only restrictions on the derivation are those generally assumed for parallel plane electrodes, negligible initial velocities of emission, and an evacuated space.

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LIST OF SYMBOLS

A ... constant

C ... constant

C_1 ... constant

C_2 ... constant

C_3 ... constant

d ... spacing between emitter and collector

∇ ... operator(del)

∇^2 ... Laplacian operator(del squared)

e ... magnitude of electronic charge unit

ϵ_0 ... permittivity of free space

E ... electric field intensity

E ... complete elliptic integral of the second kind

f ... Nordheim elliptic function

i ... imaginary unit

J ... current density

k ... constant

K ... proportionality constant

K ... complete elliptic integral of the first kind

K_1 ... proportionality constant

K_2 ... proportionality constant

m ... electronic mass

m ... mass of positive charge

M ... variable

n ... number of positive charges per unit volume

N ... variable

ρ ... volume charge density of positive charge

ϕ ... work function

q ... positive charge

v ... velocity of positively charged particle

V ... electric potential

V_d ... potential of collector

V_o ... potential of emitter

V' ... magnitude of electric field intensity

V'_d ... magnitude of the electric field intensity at collector

V'_o ... magnitude of the electric field intensity at emitter

ΔV ... potential difference between emitter and collector

ω ... imaginary cube root of unity

ω^2 ... imaginary cube root of unity

x ... distance from emitter

X ... field at the emitter to field for no charge ratio

y ... variable of Nordheim elliptic function

Y ... variable

Z ... variable

bar over symbol indicates vector quantity

CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

A. Electrical Spraying.(1)

Electrical spraying is a process whereby the application of a sufficiently high electric field to the liquid emerging from a capillary tube causes the liquid to disperse into numerous small droplets which are electrically charged. Basically, the process is one that involves an instability at a liquid interface owing to electrostatic forces, produced by induced surface charges, overcoming the cohesive forces of the liquid. The requisite surface charge can be produced by allowing the liquid in question to emerge under pressure from the end of a metal capillary maintained at a positive d.c. voltage with respect to an associated ground plate or accelerating electrode. The capillary tube is usually held positive with respect to the ground plate in order to reduce field emission effects at the conducting tip. The minimum value of voltage required to initiate electrical spraying is called the minimum spraying voltage. The liquid pressure is ordinarily insufficient to cause a liquid jet in the absence of the applied voltage. Subject to the electric field at the tip of the conduction tube, a quantity of liquid forms into the approximate shape of a hemisphere before it

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Single number in parenthesis refers to reference listed in the bibliography.

moves away. By modifying the experimental arrangements properly, all of the charged droplets moving away from the ends of several capillary tubes may be considered as if they were emitted from a plane parallel to the ground plate or collector. The charges and masses of the emitted particles are clearly dependent on the electric field at the emitting surface.

B. Statement of the Problem.

One purpose of this study is to derive a general expression for the current density passing between two parallel plane electrodes (emitter and collector) as a function of "field at the emitter to field for no charge ratio". The positive charges produced by the field at the emitter and the masses of the positively charged particles are related to the electric field at the emitter under different assumptions. The current densities corresponding to each of the aforementioned assumptions are then evaluated separately. The maximum current densities for these cases are also compared numerically with that given by the Child equation.

The second purpose of this study is to solve the problem of the potential distribution for one particular case in which the positive charges are assumed to be proportional to the field at the emitter. The electric field, the velocity of charges, and the charge density for that case are then evaluated as a function of the potential and plotted as a function of the distance for one numerical example.

C. Significance of the Study.

Investigations concerning the characteristics of electrons in the plane diode have been done by many investigators. However, a complete theoretical treatment of the problem in which the charges and masses of the charged particles are dependent on the electric field at the emitter, is not known. A detailed investigation of this situation will be useful to electrical spraying problems.

D. Review of the Literature and Reasons for the Investigation.

To the author's knowledge, the only literature readily available which discusses the topic of this study is one by Barré. (2) Barré was probably the first to obtain an expression for the current density passing between two parallel plane electrodes under the condition that the positive charge is directly proportional to the electric field at the emitter. An interest in the theory and application of this situation led the author to explore this problem for a possible thesis topic.

CHAPTER II

CURRENT DENSITY PASSING BETWEEN TWO PARALLEL PLANE ELECTRODES

A. Introduction.

The characteristics of vacuum diodes under conditions of complete space-charge limiting, i.e., space-charge density so great that the electric field at the cathode is zero, were calculated long ago. The results are expressed by the familiar Child equation for plane geometry,

$$J = \frac{4}{9}\epsilon_0\left(\frac{2e}{m}\right)^{\frac{1}{2}}\frac{(\Delta V)^{3/2}}{d^2}$$

where (ΔV) is the potential difference between the emitter and the collector, ϵ_0 is the permittivity of free space, e is the electronic charge, m, is the electronic mass, d is the spacing between the electrodes, and J indicates current density. The problem of the diode with field-dependent positive charges rather than electrons under space-charge conditions has, however, received little attention. This chapter presents a solution of this problem.

The discussion is restricted to cases where initial velocities of emission can be neglected. The emitter is also assumed to be equipotential and ideally smooth (i.e., the surface roughness is neglected).

B. General Solution for the Current Density Passing Between Two Parallel Plane Electrodes.

Consider two parallel plane electrodes(emitter and collector) separated by a distance d. Let the distance x be measured from the emitter and let the space potential V satisfy the boundary conditions $V = V_o$ at $x = 0$ and $V = V_d$ at $x = d$ with $V_o > V_d \geqq 0$. This means the emitter is maintained at a potential more positive than the collector. This situation is shown in Fig. 2-1.

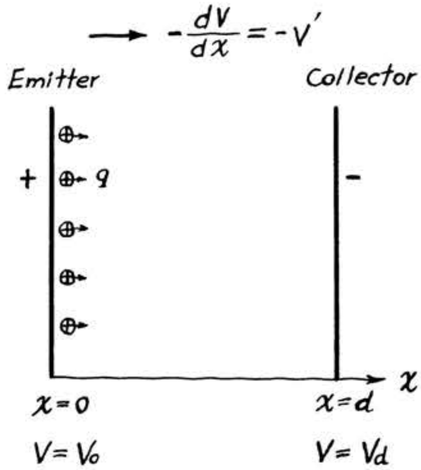

Fig. 2-1. Two parallel plane electrodes.

The distribution of the potential in the space between the electrodes is determined by Poisson's equation, which in this case involves only the coordinates x,

$$\nabla^2 V = d^2V/dx^2 = -\frac{\rho}{\epsilon_o} \tag{2-1}$$

where ρ is the positive charge space-charge volume density. As auxiliary relations one has also the current flow equation,

$$\bar{J} = \rho \bar{v} \tag{2-2}$$

and the conservation of energy equation,

$$mv^2/2 = q(V_o - V) , \tag{2-3}$$

where m is the mass of the positively charged particles, V_o is the emitter voltage, v is the velocity of the positive charges at the point x and the initial velocities of the emission have been assumed negligible.

Substituting Eq.(2-2) and $\rho = nq$ into $\nabla \cdot \bar{J} = 0$ yields

$$q\nabla \cdot n\bar{v} = 0.$$

Therefore

$$nv = A \tag{2-4}$$

and

$$J = qnv = qA, \tag{2-5}$$

where n is the number of positive charges per unit volume, A is a positive constant, $\bar{J}$ and $\bar{v}$ are the vector notations for the current density and the velocity of the positive charges, respectively.

With the aid of these equations ρ and v can be eliminated to yield

$$\frac{d^2V}{dx^2} = -k(V_o - V)^{-\frac{1}{2}}, \tag{2-6}$$

where

$$k = \frac{A}{\epsilon_o}\left(\frac{qm}{2}\right)^{\frac{1}{2}} \tag{2-7}$$

is a constant independent of x.

If Eq.(2-6) is multiplied by the factor 2(dV/dx) it may be integrated to give

$$\left(\frac{dV}{dx}\right)^2 = 4k(V_o - V)^{\frac{1}{2}} + C_1. \tag{2-8}$$

The magnitude of the electric field, dV/dx or V', at the emitter ($x = 0$, $V = V_o$) is designated by V_o'. Using this as a boundary condition, $C_1 = V_o'^2$, and one can rewrite Eq.(2-8) as

$$V'^2 - V_o'^2 = 4k(V_o - V)^{\frac{1}{2}}. \tag{2-9}$$

The magnitude of the electric field at the collector ($x = d$, $V = V_d$) is designated by V_d'. Using this condition, Eq.(2-9) yields

$$V_d'^2 - V_o'^2 = 4k(V_o - V_d)^{\frac{1}{2}} \tag{2-10}$$

or
$$V_d'^2 - V_o'^2 = 4k(\Delta V)^{\frac{1}{2}}, \tag{2-11}$$

where $\Delta V = V_o - V_d$ is the potential difference between the emitter and the collector.

Multiply the left hand side of Eq.(2-9) by dV' and the right hand side by the equivalent quantity

$$dV' = \frac{d^2V}{dx^2}\, dx = -\, k(V_o - V)^{-\frac{1}{2}}dx,$$

to obtain

$$V'^2dV' - V_o'^2dV' = -\, 4k^2dx.$$

This can be integrated

$$\int_{V_o'}^{V_d'} V'^2dV' - \int_{V_o'}^{V_d'} V_o'^2dV' = -\, 4k^2 \int_0^d dx$$

to give

$$V_d'^3 - V_o'^3 - 3V_o'^2(V_d' - V_o') = -\, 12k^2d. \tag{2-12}$$

Eliminating k between Eqs.(2-11) and (2-12) yields

$$V_d'^2 + V_d'V_o' + V_o'^2 - 3V_o'^2 + \frac{3d}{4(\Delta V)}(V_d' - V_o')(V_d' + V_o')^2 = 0. \tag{2-13}$$

On the left hand side of Eq.(2-13) add and subtract

$3V_d'V_o'$. Finally one obtains a quadratic equation in V_d' as

$$3V_d'^2 + 2(2\frac{\Delta V}{d} + 3V_o')V_d' + V_o'(8\frac{\Delta V}{d} + 3V_o') = 0. \quad (2\text{-}14)$$

This equation can be solved for V_d' to give

$$V_d' = -\frac{1}{3}(2\frac{\Delta V}{d} + 3V_o') \pm \frac{1}{3}\left[(2\frac{\Delta V}{d} + 3V_o')^2 - 3(8\frac{\Delta V}{d} + 3V_o')V_o'\right]^{\frac{1}{2}} \quad (2\text{-}15)$$

or

$$(-V_d') = \frac{\Delta V}{d}\left[\frac{2}{3} - \frac{(-V_o')}{\Delta V/d} \mp \frac{2}{3}\left\{1 + 3\frac{(-V_o')}{\Delta V/d}\right\}^{\frac{1}{2}}\right], \quad (2\text{-}16)$$

Here the positive sign before the square root is to be used*. Therefore, the electric field at the collector is given as

$$(-V_d') = \frac{\Delta V}{d}\left[\frac{2}{3} - \frac{(-V_o')}{\Delta V/d} + \frac{2}{3}\left\{1 + 3\frac{(-V_o')}{\Delta V/d}\right\}^{\frac{1}{2}}\right], \quad (2\text{-}17)$$

Eliminating $(-V_d')$ between Eqs. (2-11) and (2-17) one obtains the relation

$$(-V_o')^2 + 4k(\Delta V)^{\frac{1}{2}} = \left(\frac{\Delta V}{d}\right)^2\left[\frac{2}{3} - \frac{(-V_o')}{\Delta V/d} + \frac{2}{3}\left\{1 + 3\frac{(-V_o')}{\Delta V/d}\right\}^{\frac{1}{2}}\right]^2. \quad (2\text{-}18)$$

Define

$$X = \frac{(-V_o')}{\Delta V/d} = \frac{\text{field at emitter}}{\text{field for no charge}}, \ (0 \leq X \leq 1). \quad (2\text{-}19)$$

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* It will be shown in Chapter III that the negative sign can not be used.

Then Eq.(2-18) becomes

$$X^2\left(\frac{\Delta V}{d}\right)^2 + 4k(\Delta V)^{\frac{1}{2}} = \left(\frac{\Delta V}{d}\right)^2\left[\frac{2}{3} - X + \frac{2}{3}(1 + 3X)^{\frac{1}{2}}\right]^2 .$$

Substituting k from Eq.(2-7) into the above equation and solving for A

$$A = \left[\frac{1}{9}\epsilon_o\left(\frac{2}{qm}\right)^{\frac{1}{2}}\frac{(\Delta V)^{3/2}}{d^2}\right]\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right] .$$

Since J = qA,

therefore

$$J = \left[\frac{1}{9}\epsilon_o\left(\frac{2q}{m}\right)^{\frac{1}{2}}\frac{(\Delta V)^{3/2}}{d^2}\right]\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right] . \qquad (2\text{-}20)$$

This is the general expression for the current density passing between two parallel plane electrodes.

It is of interest also to compare the current density of Eq.(2-20) with that by the Child equation. Rewriting Eq.(2-20) in terms of the Child's law value

$$J = \frac{1}{4}\left[\frac{4}{9}\epsilon_o\left(\frac{2q}{m}\right)^{\frac{1}{2}}\frac{(\Delta V)^{3/2}}{d^2}\right]\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right]$$

or

$$J = \frac{1}{4}\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right]\left[\text{Child's law value}\right] , \qquad (2\text{-}21)$$

where Child's law value is given by

$$\frac{4}{9}\epsilon_o\left(\frac{2q}{m}\right)^{\frac{1}{2}}\frac{(\Delta V)^{3/2}}{d^2} .$$

C. Solutions of Current Density Under Different Assumptions of Positive Charges.

At this point four cases will be discussed according to different assumptions pertaining to the positive charges.

1. Case 1.

Assume the electric field produces the positive charge

$$q = KE_o = K(-V_o') \tag{2-22}$$

at the positive electrode(emitter), where K is the proportionality constant, and $E_o = -V_o'$ is the electric field at the emitter. The mass m is constant and independent of the electric field.

Substituting Eq.(2-22) into Eq.(2-20) yields

$$J = \left[\frac{1}{9}\epsilon_o \left(\frac{2K}{m}\right)^{\frac{1}{2}} \frac{(\Delta V)^2}{d^{5/2}} \right] \sqrt{X} \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right]$$

or $$\frac{J}{C} = \sqrt{X} \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right], \tag{2-23}$$

where $C = \frac{1}{9}\epsilon_o \left(\frac{2K}{m}\right)^{\frac{1}{2}} \frac{(\Delta V)^2}{d^{5/2}}$.

By differentiating Eq.(2-23), the maximum value is found to be

$$\frac{J_{max}}{C} = 2.02 \tag{2-24}$$

at $X = 0.42$.

By means of Eq.(2-21)

$$J_{max} = 0.78 \left[\text{ Child's law value } \right]. \tag{2-25}$$

The same expression for J_{max} calculated by Barré[2] was

$$J_{max} = 0.72 \left[\text{ Child's law value } \right]$$

with a factor different from what the author has obtained in Eq.(2-25).

The current density expressed by Eq.(2-23) is plotted in Fig. 2-2 as a function of the parameter X.

It is seen that the current density increases more rapidly in the region where the field at the emitter is much weaker than the field in the absence of space-charge.

In the next chapter an analytical expression for the potential will be derived for this case.

2. Case 2.

Assume the electric field produces the positive charge

$$q = K_1 E_o = K_1(-V_o') \tag{2-26}$$

at the emitter, and the mass, m, of the positively charged particle is directly proportional to the field at the emitter

$$m = K_2 E_o = K_2(-V_o'), \tag{2-27}$$

where K_1 and K_2 are proportionality constants.

Substituting Eqs.(2-26) and (2-27) into Eq.(2-20)

$$J = \left[\frac{1}{9} \epsilon_0 \left(\frac{2K_1}{K_2} \right)^{\frac{1}{2}} \frac{(\Delta V)^{3/2}}{d^2} \right] \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right]$$

or $$\frac{J}{C} = 2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}, \tag{2-28}$$

where $$C = \frac{1}{9} \epsilon_0 \left(\frac{2K_1}{K_2} \right)^{\frac{1}{2}} \frac{(\Delta V)^{3/2}}{d^2} \quad .$$

Eq.(2-28) is plotted in Fig. 2-3 as a monotonic

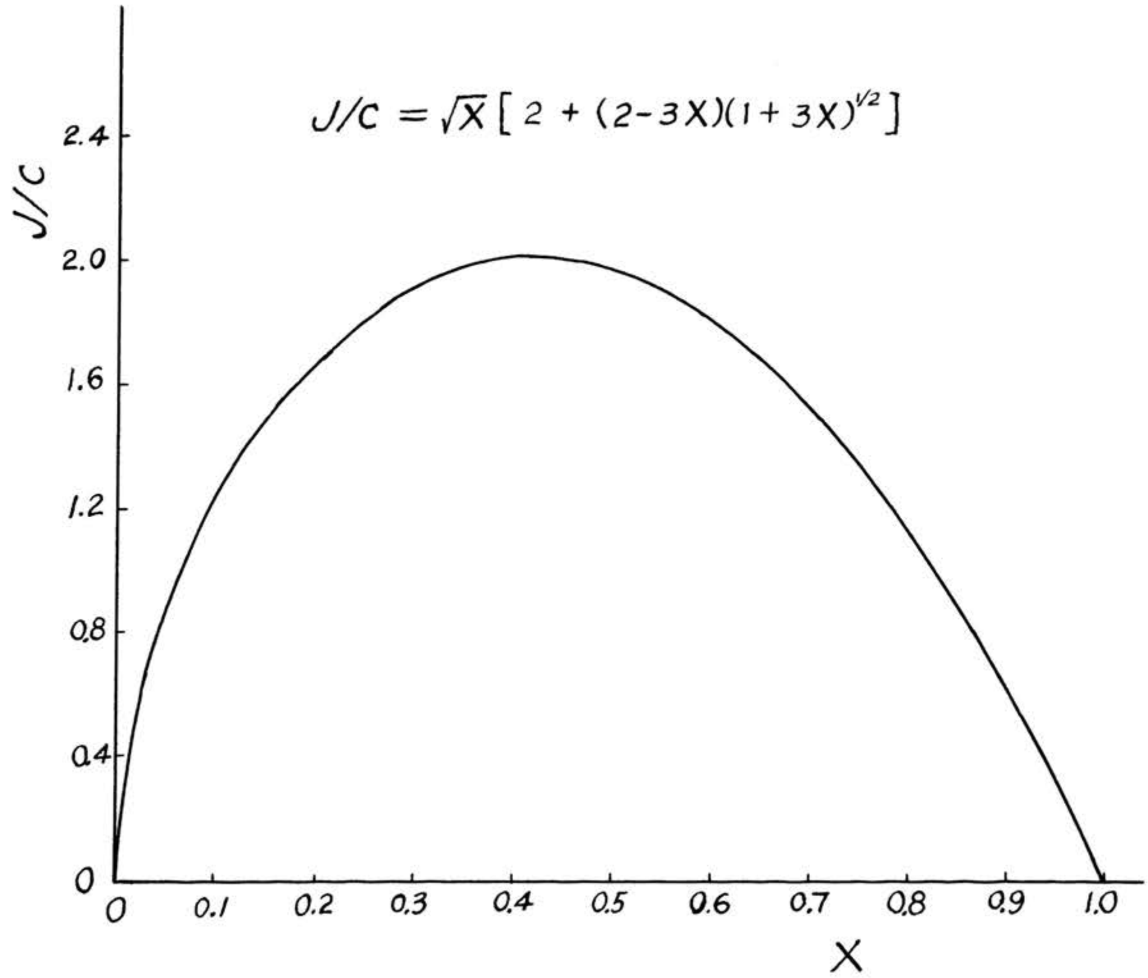

Fig. 2-2. Current density distribution of Case 1.

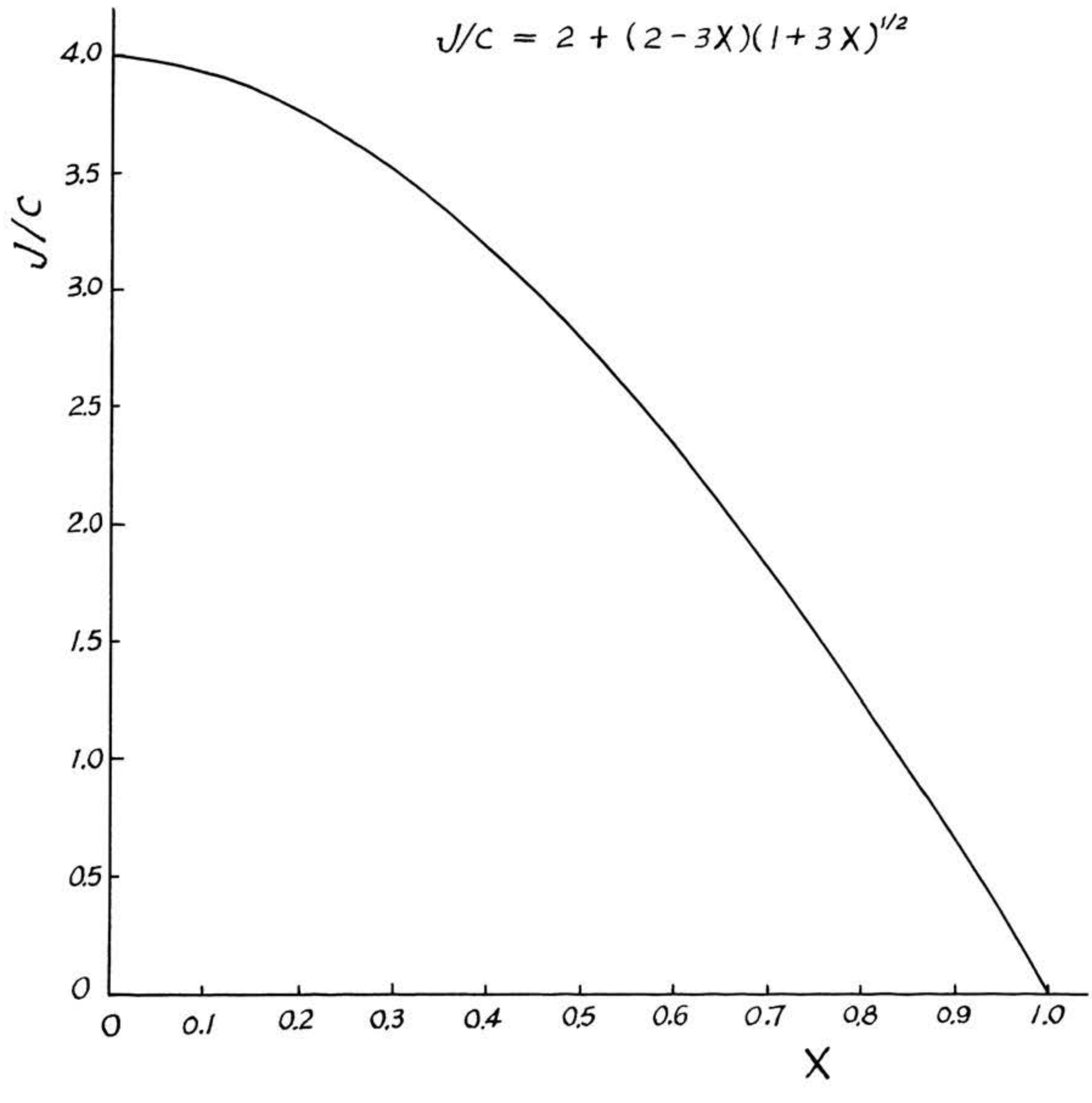

Fig. 2-3. Current density distribution of Case 2.

decreasing curve with a maximum of

$$\frac{J_{max}}{C} = 4 \tag{2-29}$$

at X = 0.

From Eq.(2-21) it is of interest to note that the maximum current density

$$J_{max} = \left[\text{Child's law value} \right] \tag{2-30}$$

is the same as the Child's law value.

3. Case 3.

Assume the electric field produces the positive charge

$$q = K_1 E_o = K_1(-V_o') \tag{2-31}$$

at the emitter, and the mass, m, of the positively charged particle is inversely proportional to the field at the emitter

$$m = \frac{K_2}{E_o} = \frac{K_2}{(-V_o')} , \tag{2-32}$$

where K_1 and K_2 are proportionality constants.

Substituting Eqs.(2-31) and (2-32) into Eq.(2-20)

$$J = \left[\frac{1}{9}\epsilon_o \left(\frac{2K_1}{K_2}\right)^{\frac{1}{2}} \frac{(\Delta V)^{5/2}}{d^3} \right] X \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right]$$

or

$$\frac{J}{C} = X \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right] , \tag{2-33}$$

where

$$C = \frac{1}{9}\epsilon_o \left(\frac{2K_1}{K_2}\right)^{\frac{1}{2}} \frac{(\Delta V)^{5/2}}{d^3} .$$

Eq.(2-33) is plotted in Fig. 2-4. The maximum value

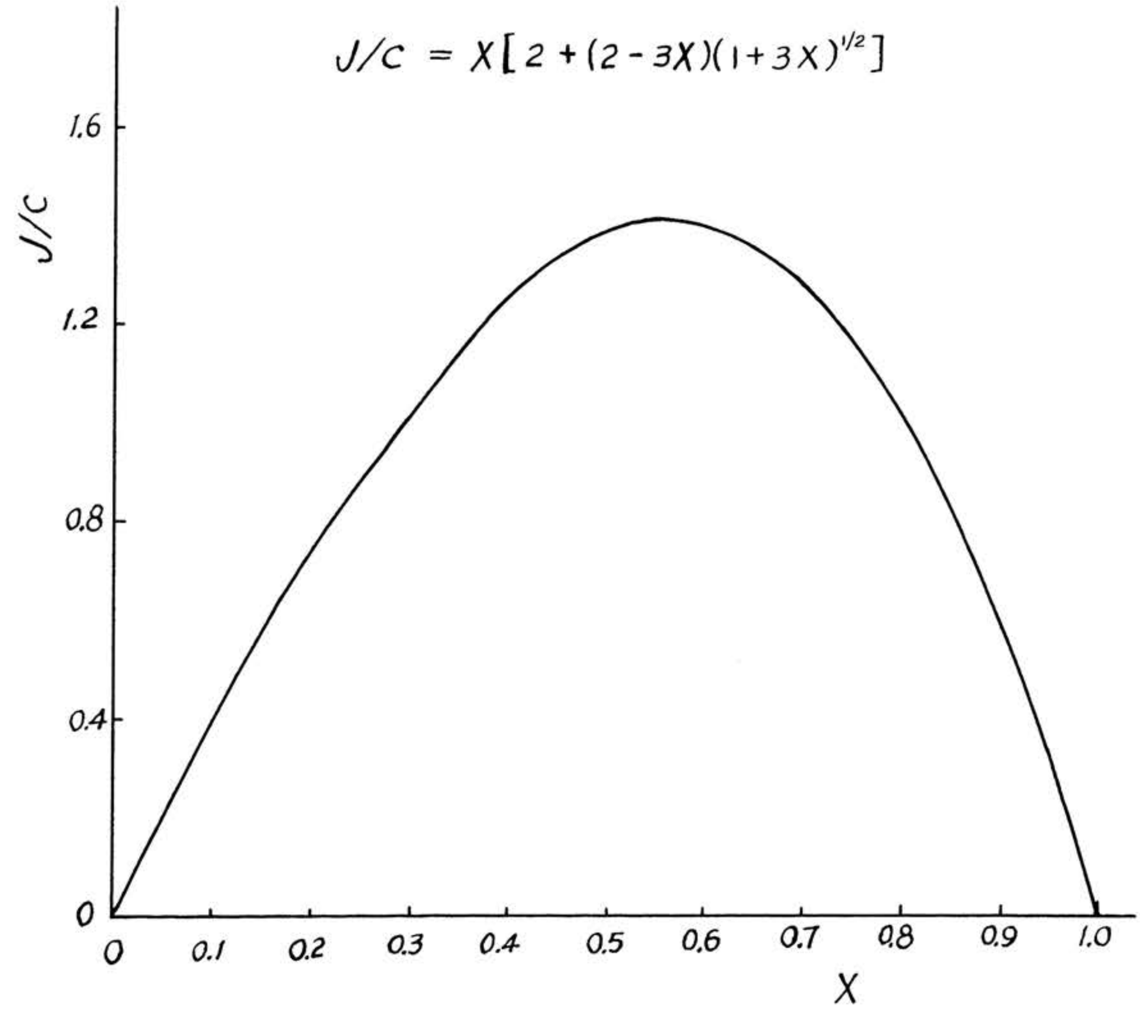

Fig. 2-4. Current density distribution of Case 3.

is found to be

$$\frac{J_{max}}{C} = 1.42 \qquad (2\text{-}34)$$

at X = 0.56.

From Eq.(2-21) the maximum current density is found to be

$$J_{max} = 0.63 \left[\text{Child's law value} \right]. \qquad (2\text{-}35)$$

In Fig. 2-4, it can be seen that the current density increases more slowly in the region where the field at the emitter is much weaker than the field in the absence of space-charge.

4. Case 4.

In this final case, assume the electric field produces the positively charged particle with a "charge-to-mass ratio" proportional to the field at the emitter*,

$$\frac{q}{m} = KE_o = K(-V_o'), \qquad (2\text{-}36)$$

where K is the proportionality constant.

Substituting Eq.(2-36) into Eq.(2-20)

$$J = \left[\frac{1}{9} \epsilon_o (2K)^{\frac{1}{2}} \frac{(\Delta V)^2}{d^{5/2}} \right] \sqrt{X} \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right]$$

or

$$\frac{J}{C} = \sqrt{X} \left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}} \right], \qquad (2\text{-}37)$$

where

$$C = \frac{1}{9} \epsilon_o (2K)^{\frac{1}{2}} \frac{(\Delta V)^2}{d^{5/2}}.$$

---

* Mathematically, there is no difference between Case 4 and Case 1. The results of Case 4 may be obtained readily by subtituting Km for K in Case 1.

Eq.(2-37) is exactly the same as Eq.(2-23) except that the C's are different. Therefore, the curve plotted for Eq.(2-37) will be the same as Fig. 2-2. As calculated in Case 1, $X \approx 0.42$ makes a maximum of

$$\frac{J_{max}}{C} = 2.02$$

and

$$J_{max} = 0.78 \left[\text{Child's law value} \right].$$

5. Summary.

The previous discussions for the different cases may be summarized in tabular form as TABLE 2-1.

TABLE 2-1

SUMMARY OF CASES

Cases	Assumptions*	Constants C	Expressions of J/C	$\frac{J_{max}}{C}$	Comparison of J_{max} with Child's law value
1	$q = KE_0$ $m = \text{constant}$	$\frac{1}{9}\epsilon_0\left(\frac{2K}{m}\right)^{1/2}\frac{(\Delta V)^2}{d^{5/2}}$	$\sqrt{X}\left[2+(2-3X)(1+3X)^{1/2}\right]$	2.02	0.78 [Child's law value]
2	$q = K_1E_0$ $m = K_2E_0$	$\frac{1}{9}\epsilon_0\left(\frac{2K_1}{K_2}\right)^{1/2}\frac{(\Delta V)^{3/2}}{d^2}$	$2+(2-3X)(1+3X)^{1/2}$	4.00	[Child's law value]
3	$q = K_1E_0$ $m = K_2/E_0$	$\frac{1}{9}\epsilon_0\left(\frac{2K_1}{K_2}\right)^{1/2}\frac{(\Delta V)^{5/2}}{d^3}$	$X\left[2+(2-3X)(1+3X)^{1/2}\right]$	1.42	0.63 [Child's law value]
4	$\frac{q}{m} = KE_0$	$\frac{1}{9}\epsilon_0(2K)^{1/2}\frac{(\Delta V)^2}{d^{5/2}}$	$\sqrt{X}\left[2+(2-3X)(1+3X)^{1/2}\right]$	2.02	0.78 [Child's law value]

* These refer to the particular assumptions for each case in addition to the general assumptions as stated in the very beginning of this chapter.

CHAPTER III

POTENTIAL DISTRIBUTION BETWEEN TWO PARALLEL PLANE ELECTRODES

A. Derivation and Discussions of the Potential Distribution Between Two Parallel Plane Electrodes.

In this chapter, the complete potential distribution between two parallel plane electrodes is to be investigated for Case 1 as already mentioned in Chapter II. In this case, it was assumed that the electric field produced the positive charge

$$q = K(-V_o') \tag{3-1}$$

at the emitter. As derived in Chapter II, Eq.(2-9) can be rewritten as

$$V'^2 - V_o'^2 = 4k(V_o - V)^{\frac{1}{2}}. \tag{3-2}$$

Solving for V', i.e., dV/dx

$$\frac{dV}{dx} = \pm\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}} . \tag{3-3}$$

Since the potential distribution is a monotonic decreasing curve*, the negative sign is used and the square root quantity is considered as positive.

Thus

$$\frac{dV}{dx} = -\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}} . \tag{3-4}$$

---

* If one sets Eq.(3-3) to be zero, no point in the range between x=0 and x=d will yield a slope of zero value.

Eq.(3-4) can be rewritten as

$$\frac{2(V_o - V)^{\frac{1}{2}}\, d(V_o - V)^{\frac{1}{2}}}{\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}}} = dx \,.$$

This can be integrated to give

$$\left[2k(V_o - V)^{\frac{1}{2}} - V_o'^2\right]\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 6k^2x + C_2 \,.$$

The constant of integration can be determined from the requirement that $V = V_o$ at $x = 0$, or $C_2 = V_o'^3$.

Therefore

$$\left[2k(V_o - V)^{\frac{1}{2}} - V_o'^2\right]\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 6k^2x + V_o'^3. \qquad (3\text{-}5)$$

Eq.(3-5) must also be satisfied by the boundary condition at the collector that $V = V_d$ at $x = d$. Substituting this condition and $V_o - V_d = \Delta V$, Eq.(3-5) becomes

$$\left[2k(\Delta V)^{\frac{1}{2}} - V_o'^2\right]\left[V_o'^2 + 4k(\Delta V)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 6k^2d + V_o'^3.$$

Squaring both sides and simplifying, the above equation reduces to a cubic equation in V_o',

$$V_o'^3 + \left(\frac{\Delta V}{d}\right) V_o'^2 + \frac{1}{3d}\left[9k^2d^2 - 4k(\Delta V)^{3/2}\right] = 0. \qquad (3\text{-}6)$$

By Cardan's formula*, Eq.(3-6) is solved for V_o' and the three cube roots are

$$V_o' = Y + Z - \frac{1}{3}\left(\frac{\Delta V}{d}\right), \qquad (3\text{-}7)$$

$$V_o' = \omega Y + \omega^2 Z - \frac{1}{3}\left(\frac{\Delta V}{d}\right), \qquad (3\text{-}8)$$

and $$V_o' = \omega^2 Y + \omega Z - \frac{1}{3}\left(\frac{\Delta V}{d}\right), \qquad (3\text{-}9)$$

---

* See Appendix.

where

$$Y = \left[-\frac{N}{2} + \frac{1}{2}(N^2 + \frac{4}{27}M^3)^{\frac{1}{2}} \right]^{1/3}, \quad (3\text{-}10)$$

$$Z = \left[-\frac{N}{2} - \frac{1}{2}(N^2 + \frac{4}{27}M^3)^{\frac{1}{2}} \right]^{1/3}, \quad (3\text{-}11)$$

$$N = -\frac{1}{3d}\left[4k(\Delta V)^{3/2} - 9k^2d^2\right] + \frac{2}{27}(\frac{\Delta V}{d})^3, \quad (3\text{-}12)$$

$$M = -\frac{1}{3}(\frac{\Delta V}{d})^2, \quad (3\text{-}13)$$

$$N^2 + \frac{4}{27}M^3 = \frac{k}{d^2}\left[9d^2k - 4(\Delta V)^{3/2}\right]\left[dk - \frac{2(\Delta V)^{3/2}}{9\ d}\right]^2, \ k > 0. \quad (3\text{-}14)$$

and

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\ ;\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\ ;\ 1 + \omega + \omega^2 = 0, \quad (3\text{-}15)$$

where ω and ω^2 are the imaginary cube roots of unity.

Under the constraint that the electric field ($-V_o'$) at the emitter must be a non-zero, positive real value, the roots for V_o' can be discussed as follows:

1. When

$$N^2 + \frac{4}{27}M^3 > 0,$$

then

$$\left[9d^2k - 4(\Delta V)^{3/2}\right]\left[dk - \frac{2}{9}\frac{(\Delta V)^{3/2}}{d}\right]^2 > 0.$$

Solving the above inequality for k,

$$k > \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2},$$

i.e., $X > 1$. The positive, non-zero values of current densities can not be obtained. Furthermore, the boundary condition at the collector will not be satisfied.

2. When

$$N^2 + \frac{4}{27}M^3 = 0,$$

then

$$\left[9d^2k - 4(\Delta V)^{3/2}\right]\left[\ dk - \frac{2}{9}\frac{(\Delta V)^{3/2}}{d}\right]^2 = 0.$$

Solving the above equation for k,

$$k = \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2} \quad ,$$

or $$k = \frac{2}{9}\frac{(\Delta V)^{3/2}}{d^2} \quad .$$

For the value of $k = \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2}$; X = 1 and X = 0 and the positive, non-zero values of current densities can not be obtained. Furthermore, the boundary condition at the collector will not be satisfied. For the value of $k = \frac{2}{9}\frac{(\Delta V)^{3/2}}{d^2}$, by Eq.(3-7), $- V_o' = -\frac{1}{3}(\frac{\Delta V}{d})$, but the electric field $(- V_o')$ at the emitter must be positive. This means that Eq.(3-7) is not the required solution of V_o'. Then either by Eq.(3-8) or Eq.(3-9), $- V_o' = \frac{2}{3}(\frac{\Delta V}{d})$ (or $X = \frac{2}{3}$). With this value of electric field at the emitter and $k = \frac{2}{9}\frac{(\Delta V)^{3/2}}{d^2}$, the boundary condition at the collector ($V = V_d$ at $x = d$) will be satisfied. This can be checked by Eq.(3-5).

3. When

$$N^2 + \frac{4}{27}M^3 < 0,$$

then $$\left[9d^2k - 4(\Delta V)^{3/2}\right]\left[dk - \frac{2}{9}\frac{(\Delta V)^{3/2}}{d}\right]^2 < 0.$$

Solving the above inequality for k,

$$k < \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2} \quad .$$

This condition makes Y and Z complex conjugates and then their sum is real. The non-zero, positive real value of the electric field $(-V_o')$ at the emitter can be obtained by Eq.(3-8) and Eq.(3-9) but not Eq.(3-7).

These results allow us to conclude that with the condition

$$k < \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2} , \tag{3-16}$$

the non-zero, positive real value of the electric field $(-V_o')$ at the emitter can be obtained and the boundary condition at the collector will also be satisfied.

There is another approach to check the results that one obtained in Eq.(3-16).

As defined by Eq.(2-7)

$$k = \frac{A}{\epsilon_o}\left(\frac{qm}{2}\right)^{\frac{1}{2}}.$$

Substituting $A = J/q$ and $q = K(-V_o')$ into the above expression results in

$$k = \frac{J}{\epsilon_o}\left[\frac{m}{2K(-V_o')}\right]^{\frac{1}{2}}$$

or $$k = \frac{J}{\epsilon_o}\left[\frac{m}{2K(\Delta V/d)}\right]^{\frac{1}{2}} \frac{1}{\sqrt{X}} , \tag{3-17}$$

where $X = \dfrac{(-V_o')}{\Delta V/d}$ as defined by Eq. (2-19).

Solving for J from Eq.(3-17)

$$J = \epsilon_o k \left[\frac{2K}{m}\left(\frac{\Delta V}{d}\right)\right]^{\frac{1}{2}} \sqrt{X} \tag{3-18}$$

and rearranging terms

$$J = \left[\frac{1}{9}\epsilon_0\left(\frac{2K}{m}\right)^{\frac{1}{2}}\frac{(\Delta V)^2}{d^{5/2}}\right]\left[\frac{9d^2}{(\Delta V)^{3/2}}\right] k\sqrt{X}$$

or $\frac{J}{C} = C_3 k\sqrt{X},$ (3-19)

where $C_3 = \frac{9d^2}{(\Delta V)^{3/2}}$

and $C = \frac{1}{9}\epsilon_0\left(\frac{2K}{m}\right)^{\frac{1}{2}} \frac{(\Delta V)^2}{d^{5/2}}$ as defined in Case 1,

Chapter II.

For the value of $k < \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2}$; $C_3 k < 4$,

$$\frac{J}{C} < 4\sqrt{X}$$

The function

$$\frac{J}{C} = C_3 k\sqrt{X} \qquad (3\text{-}19)$$

with parameter $C_3 k < 4$ and $C_3 k \geqq 4$ is plotted together with the function

$$\frac{J}{C} = \sqrt{X}\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right] \qquad (2\text{-}23)$$

in Fig. 3-1.

It can be seen from Fig. 3-1 that only the curve with the parameter $C_3 k < 4$ will make an intersection with the curve $J/C = \sqrt{X}\left[2 + (2 - 3X)(1 + 3X)^{\frac{1}{2}}\right]$ within the interval $0 < X < 1$. This result allows us to conclude that the non-zero, positive real values of J/C and $-V_0'$ can be obtained with the following condition

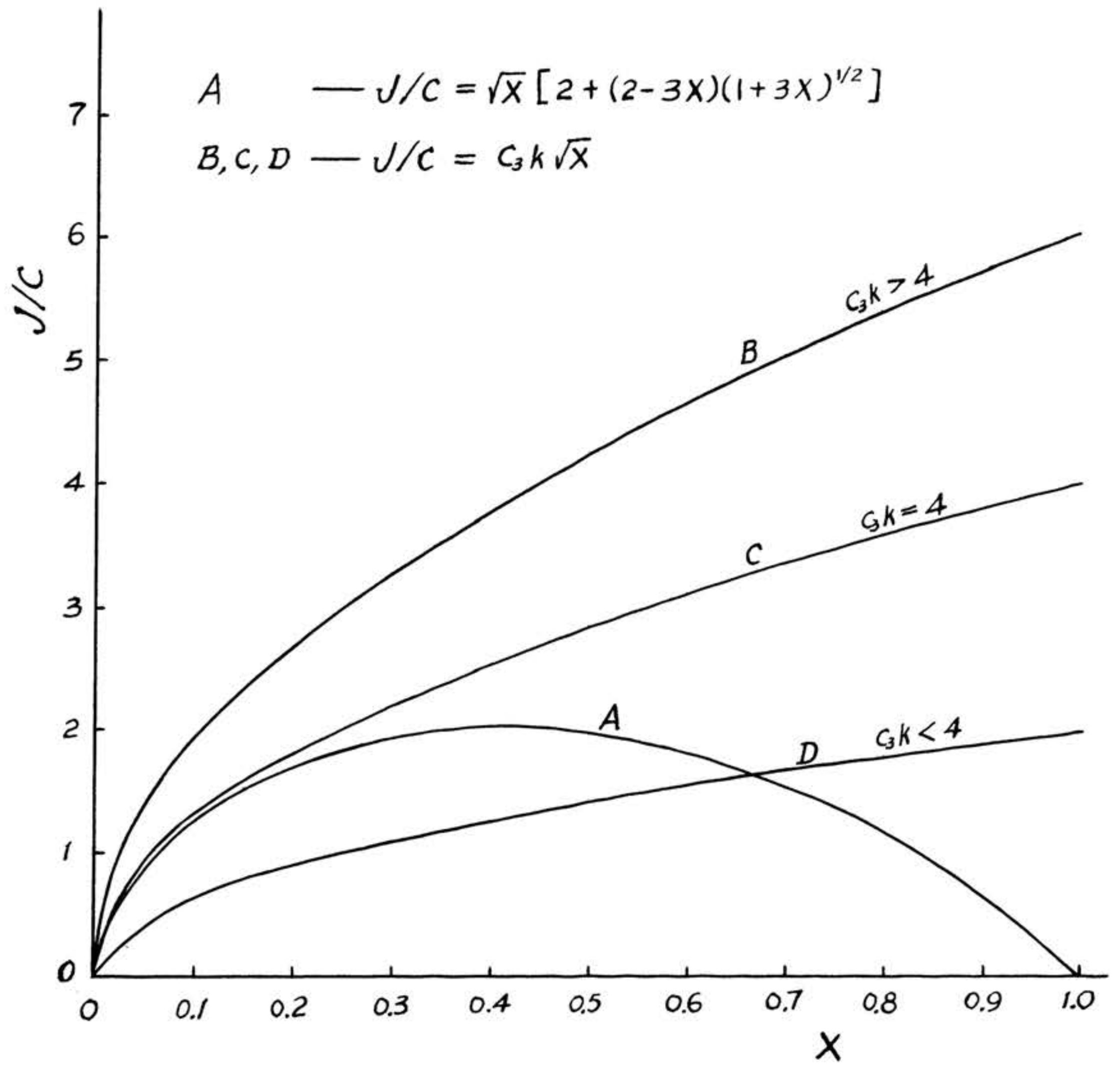

Fig. 3-1. Current density distribution curves described by Eqs.(3-19) and (2-23). The possible solution can be obtained only with $C_3k < 4$.

$$k < \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2} \;.$$

This is in agreement with the result in Eq.(3-16).

Now, let's turn back to Eq.(2-15) in Chapter II. Since $-V_o'$ is always positive, the second term under the square root is always positive. This makes $-V_d'$ negative if the minus sign before the square root were used in Eq. (2-16). Conversely, $-V_d'$ will be positive if the positive sign were used in Eq.(2-16). But $-V_d'$ has to be positive by Eq.(3-4). Therefore, the positive rather than the negative sign before the square root must be used in Eq. (2-16).

These discussions conclude that both the boundary conditions at the emitter and the collector will be satisfied by Eq.(3-5) if the condition

$$k < \frac{4}{9}\frac{(\Delta V)^{3/2}}{d^2}$$

is satisfied.

Therefore, Eq.(3-5) is the required expression which completely describes the potential distribution between the emitter and the collector. Solving for x from Eq.(3-5), one gets a different form of expression for the potential distribution,

$$x = \frac{1}{6k^2}\left\{ -V_o'^3 + \left[2k(V_o - V)^{\frac{1}{2}} - V_o'^2\right]\left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}, \quad (3\text{-}20)$$

where $\quad k = \frac{A}{\epsilon_o}\left[\frac{Km(-V_o')}{2}\right]^{\frac{1}{2}}$.

B. Fowler-Nordheim Field Emission Equation.[3]

The Fowler-Nordheim field emission equation gives a theoretical relation between the current density of field-emitted electrons and the electric field at the surface of the emitter. According to the theory the only controlling quantity that depends on the emission surface used is the work function. The Fowler-Nordheim field emission equation can be written in the following form

$$J = 1.54\times10^{-6}(E_o^2/\phi)\exp\left[-6.83\times10^7\phi^{3/2}f(y)/E_o\right], \qquad (3\text{-}21)$$

where ϕ is the work function in eV, J is the current density in amp/cm^2, E_o is the emitter surface electric field intensity in V/cm, and f(y) is the Nordheim elliptic function*[4] of the variable $y = 3.79\times10^{-4}E_o^{\frac{1}{2}}/\phi$.

---

* The Nordheim elliptic function is

$$f(y) = 2^{-\frac{1}{2}}\left[1 + (1-y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\left\{E(k^2) - y^2K(k^2)/\left[1 + (1-y^2)^{\frac{1}{2}}\right]\right\},$$

where $$E(k^2) = \int_0^{\pi/2}(1 - k^2\sin^2\phi)^{\frac{1}{2}}d\phi,$$

$$K(k^2) = \int_0^{\pi/2}(1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi,$$

$$k^2 = 2(1 - y^2)^{\frac{1}{2}}/\left[1 + (1-y^2)^{\frac{1}{2}}\right].$$

$K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kind.

It is expected that an equation similar to the Fowler-Nordheim equation can be obtained for the current density of field-emitted positive charges which are field dependent. No attempt will be made in this study to derive this equation.

The Fowler-Nordheim equation for the current density of field-emitted positive charges will be completely independent of what we have done previously. With the aid of this equation a proper operating condition can be obtained and the whole problem is then completely solved.

C. Example.

A numerical example is presented in this section. The potential distribution, electric field, velocity of charges and charge density at any point in the interelectrode space will be calculated.

Assume

$$m = 2 \times 10^{-15} \text{ kg}$$
$$K = 10^{-15} \text{ Coul-meter/Volt}$$
$$d = 10^{-2} \text{ meter}$$
$$V_o = 5000 \text{ Volts}$$
$$V_d = 0 \text{ Volt}$$
$$V = V_o - V_d = 5000 \text{ Volts.}$$

The final velocity of the positive charges is measured experimentally to be $v_d = 40.8$ kilometers/sec. Substituting this value for v and setting $V = V_d = 0$ in the relation*

$$v = \left[\frac{2K(-V_o')}{m}\right]^{\frac{1}{2}} (V_o - V)^{\frac{1}{2}},$$

* This is Eq.(3-25) to be derived later on.

$-V_o'$ is solved to be 333kV/m. By Eq.(3-6) k is found to be 7.85.

1. Potential Distribution.

Substituting the assumed as well as the calculated numerical quantities into Eq. (3-20),

$$x = \frac{1}{3.7 \times 10^{18}} \left\{ 3.7 \times 10^{16} + \left[1.57\times10^{9}(5000-V)^{\frac{1}{2}} - 11.1\times10^{10}\right] \times \left[11.1\times10^{10} + 3.14\times10^{9}(5000-V)^{\frac{1}{2}}\right]^{\frac{1}{2}} \right\} \tag{3-22}$$

By means of Eq.(3-22), the potential distribution curve is obtained as Fig. 3-2. It is noted that the potential distribution is almost a linear variation.

2. Electric Field.

The expression for the electric field can be obtained directly from Eq.(3-4),

$$E = -\frac{dV}{dx} = \left[V_o'^2 + 4k(V_o - V)^{\frac{1}{2}} \right]^{\frac{1}{2}} . \tag{3-23}$$

Substituting the numerical quantities assumed,

$$E = (10^5) \left[11.1 + 0.314(5000 - V)^{\frac{1}{2}} \right]^{\frac{1}{2}} . \tag{3-24}$$

This expresses the electric field as a function of the potential. Use of the data obtained from Fig. 3-2 yields the results shown in Fig. 3-3 which illustrates the electric field at any point in the interelectrode space. The electric field is almost a linear variation except in the region close to the emitter.

3. Velocity of Charges.

The velocity of the positive charges at any point in

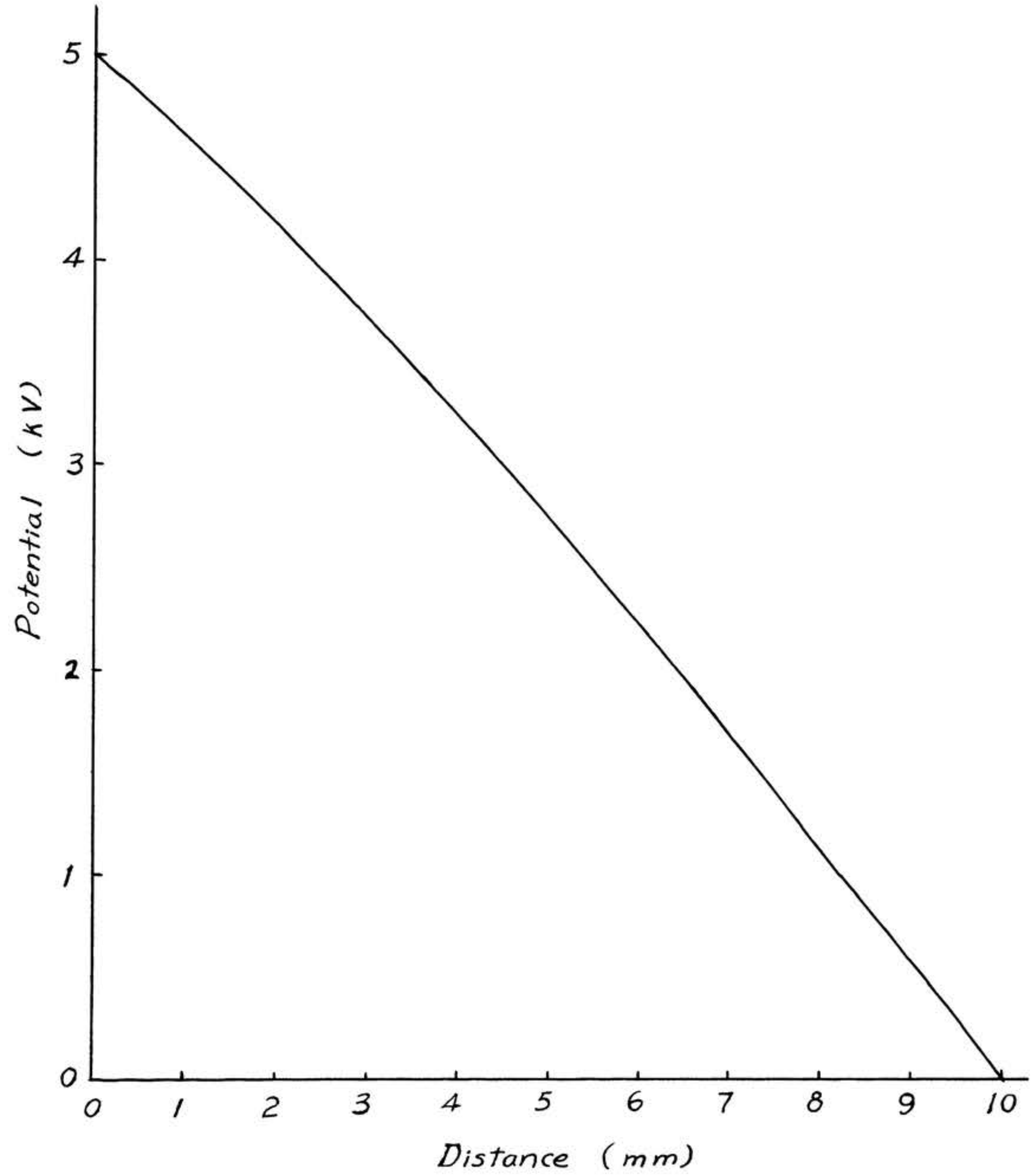

Fig. 3-2. Potential distribution between electrodes.

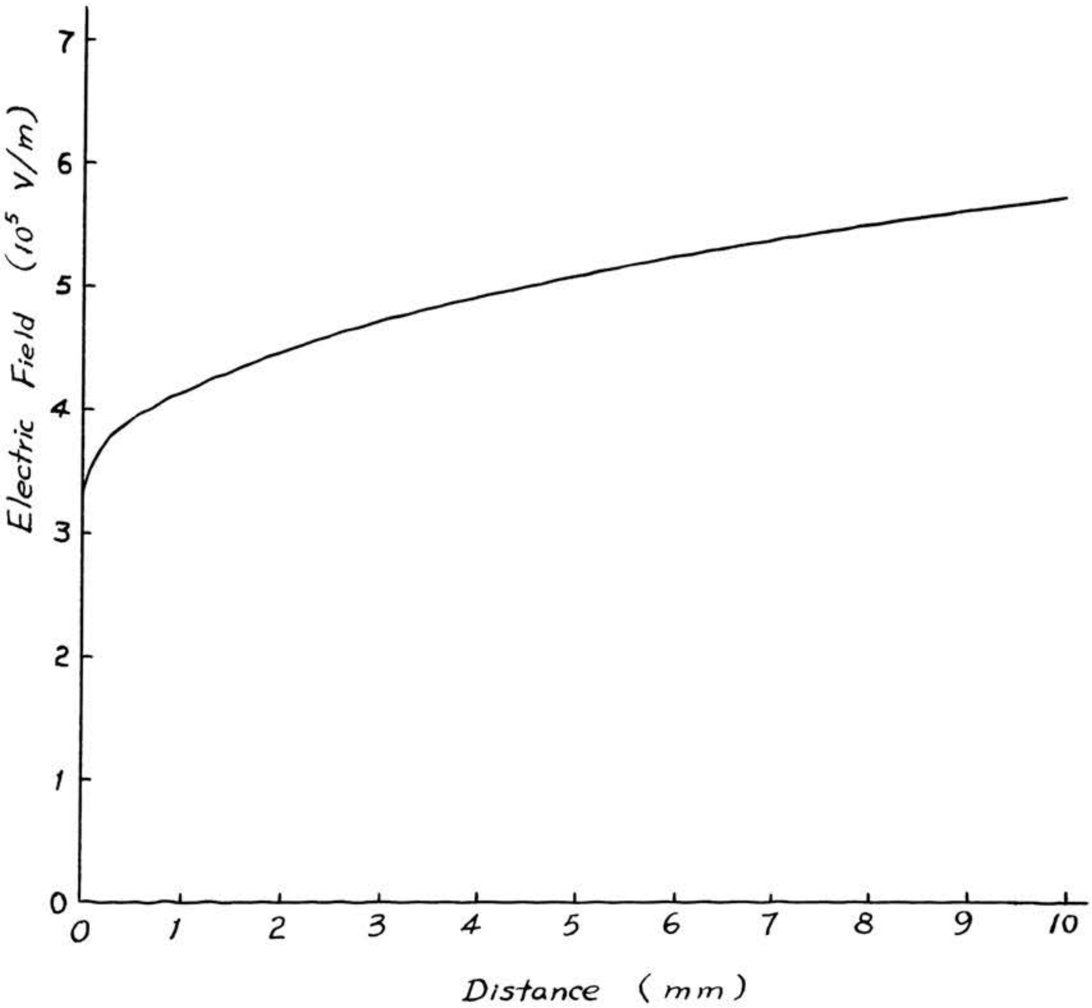

Fig. 3-3. Electric field between electrodes.

the interelectrode space may be determined from the equation that relates the kinetic energy of the particle with the potential through which it has fallen.

Assume the initial velocities of emission can be neglected. Then

$$\frac{1}{2}\,mv^2 = q(\ V_o - V\ ).$$

Solving for v and substituting $K(-V_o')$ for q,

$$v = \left[\frac{2K(-V_o')}{m}\right]^{\frac{1}{2}} (\ V_o - V\ )^{\frac{1}{2}}. \qquad (3\text{-}25)$$

Substituting the numerical quantities assumed before

$$v = 577(\ 5000 - V\ )^{\frac{1}{2}}. \qquad (3\text{-}26)$$

This expresses the velocity of the positive charges as a function of the potential. Use of the data obtained from Fig. 3-2 yields the results shown in Fig. 3-4 which illustrates the velocity of the positive charges at any point in the interelectrode space.

It can be seen that Fig. 3-4 is a parabolic curve.

4. Charge Density.

In order to get an expression for the charge density, eliminate the constant A between Eqs.(2-4) and (2-7)

$$n = \frac{k\,\epsilon_o}{v}\left(\frac{2}{qm}\right)^{\frac{1}{2}}. \qquad (3\text{-}27)$$

Substituting Eq.(3-27) and the Eq.(3-25) into the relation $\rho = nq$ to get

$$\rho = k\,\epsilon_o(\ V_o - V\ )^{-\frac{1}{2}}, \qquad (3\text{-}28)$$

where $\epsilon_o = 10^{-9}/36\pi \simeq 8.85 \times 10^{-12}$ F/m is the permittivity

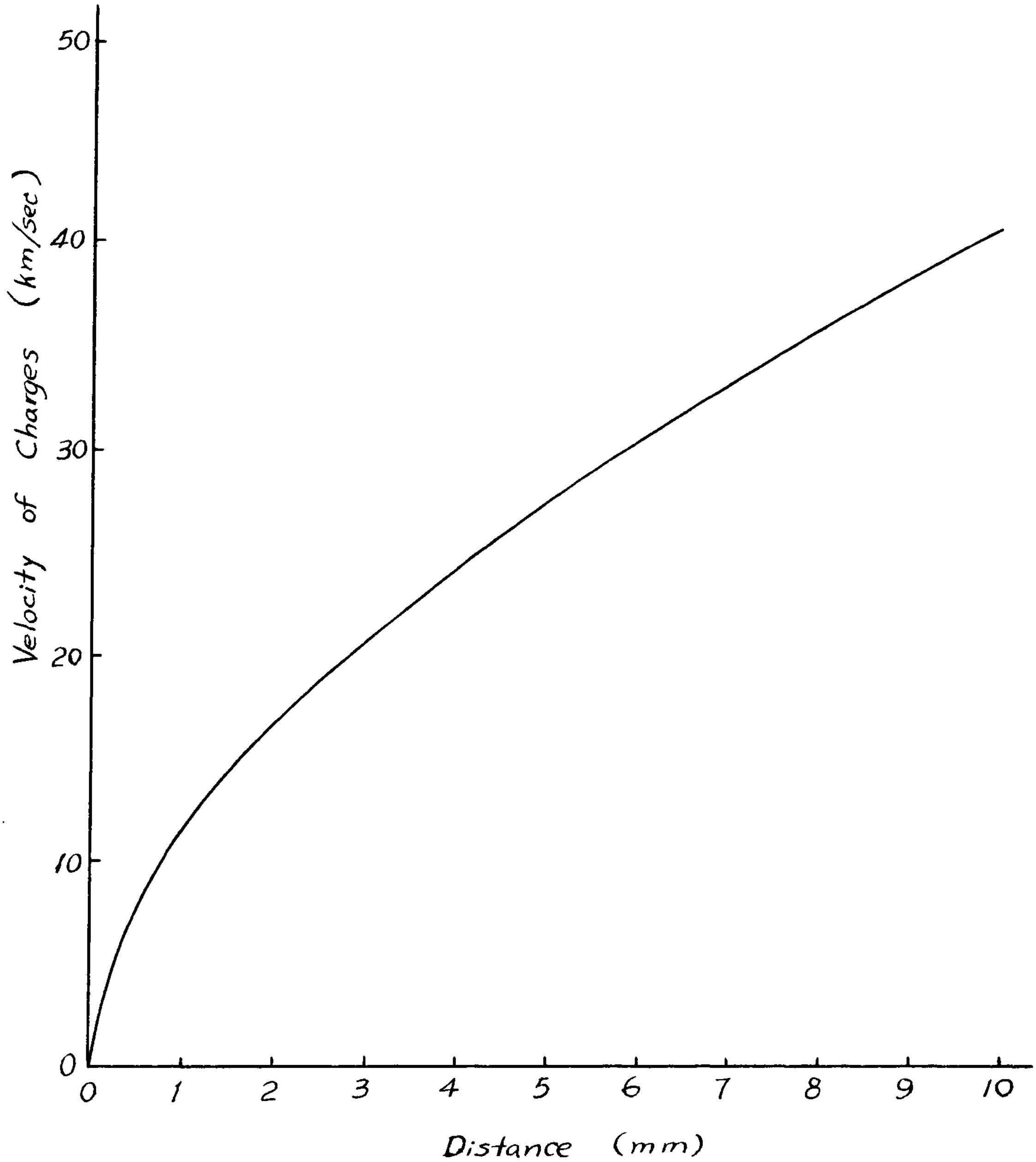

Fig. 3-4. Velocity of the positive charges between electrodes.

of free space.

Substituting the numerical quantities assumed into Eq.(3-28)

$$\rho = (6.95 \times 10^{-3})(5000 - V)^{-\frac{1}{2}}. \qquad (3\text{-}29)$$

This gives an expression for the charge density as a function of the potential. Again use of the data obtained from Fig. 3-2 yields the results shown in Fig. 3-5 which illustrates the charge density of the positive charges at any point in the interelectrode space.

As shown in Fig. 3-4 and Fig. 3-5, both ρ and v are functions of the distance from the emitter. However, their product is constant ($J = \rho v$). Therefore, near the emitter where the velocity of the charges is very small, the charge density is very large. In the neighborhood of the collector, the velocity is a maximum, hence the charge density is a minimum. This can also be seen from Fig. 3-5. This leads to the physically impossible result that at the emitter the charge density is infinite. This is a consequence of the assumption that the positive charges emerging from the emitter all do so with zero initial velocity. Actually, of course, the initial velocities are small, but finite, and the charge density is large, though finite.

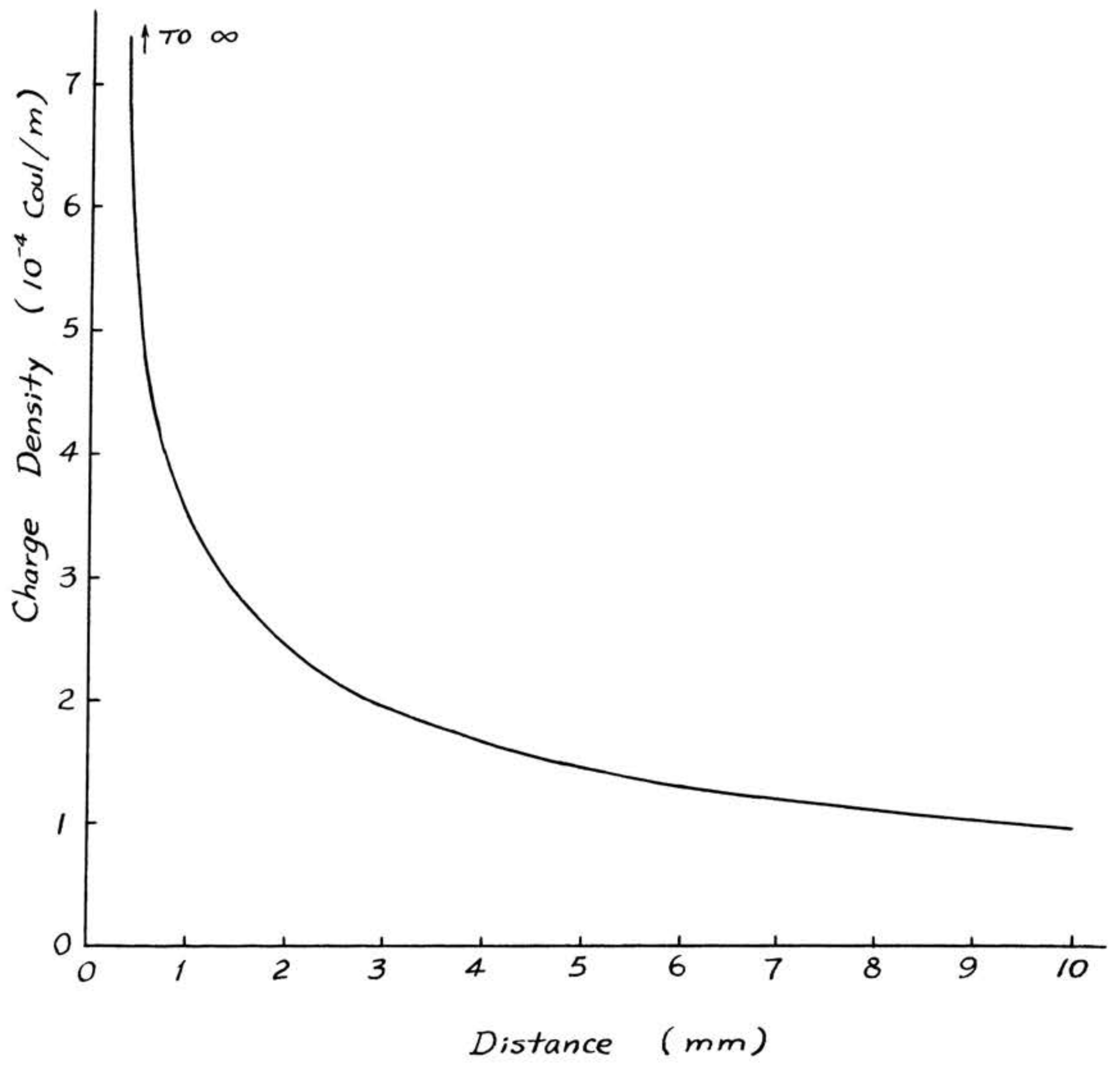

Fig. 3-5. Charge density between electrodes.

CHAPTER IV

CONCLUSIONS

The general expression for the current density passing between two parallel plane electrodes is obtained in Eq.(2-20). The current densities under different assumptions of the positive charges are then solved. The potential distribution between the emitter and the collector has been completely solved for Case 1 only. It is found that both the boundary conditions at the emitter and the collector are satisfied and a positive, non-zero value of current density is obtained if the condition

$$k < \frac{4}{9} \frac{(\Delta V)^{3/2}}{d^2} \qquad (3\text{-}16)$$

is satisfied. Under the numerical operating conditions assumed, the current density value is found to be 4×10^{10} amp/m^2. The potential distribution, the electric field, the velocity of charges and the charge density at any point in the interelectrode space have been demonstrated by a numerical example. There is no attempt to give the similar derivations and discussions for Cases 2, 3 and 4, but it is believed that there will be no difficulty in solving these cases.

It is suggested by the author that Eq.(3-5) may be solved for the potential V as a function of the distance x, so that the electric field, the velocity of charges and the charge density can be obtained from V more straightforwardly.

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APPENDIX

CARDAN'S FORMULAS FOR CUBIC EQUATIONS

The general cubic equation is

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad a_0 \neq 0. \tag{1}$$

or $$x^3 + bx^2 + cx + d = 0, \tag{2}$$

where $$b = a_1/a_0, \quad c = a_2/a_0, \quad d = a_3/a_0. \tag{3}$$

Set $$x = p - \frac{b}{3} \tag{4}$$

and (2) becomes

$$p^3 + Mp + N = 0 \tag{5}$$

in which $$M = c - b^3/3, \quad N = d - bc/3 + 2b^3/27. \tag{6}$$

Equation (5) is called the reduced cubic.

To solve the reduced cubic we introduce two unknowns, y and z, whose sum is to be a root of the reduced cubic; that is, we set

$$p = y + z \tag{7}$$

and substitute in (5). The resulting equation may be placed in the form

$$y^3 + z^3 + (3yz + M)(y + z) + N = 0. \tag{8}$$

Since we have substituted two unknowns, y and z, for the single unknown p, we can impose a condition on them. If we impose the condition

$$3yz + M = 0, \tag{9}$$

equation (8) reduces to the simpler form

$$y^3 + z^3 + N = 0. \tag{10}$$

Solving (9) for z and substituting in (10), we obtain

$$y^3 - \frac{M^3}{27y^3} + N = 0, \tag{11}$$

or $$y^6 + Ny^3 - \frac{M^3}{27} = 0. \tag{12}$$

This is a quadratic in y^3, and we find

$$y^3 = -\frac{N}{2} \pm \frac{1}{2}(N^2 + 4M^3/27)^{\frac{1}{2}}. \tag{13}$$

Taking $$y^3 = -\frac{N}{2} + \frac{1}{2}(N^2 + 4M^3/27)^{\frac{1}{2}}, \tag{14}$$

we find from (10) that

$$z^3 = -\frac{N}{2} - \frac{1}{2}(N^2 + 4M^3/27)^{\frac{1}{2}}. \tag{15}$$

NOTE: If we take $y^3 = -\frac{N}{2} - \frac{1}{2}(N^2 + 4M^3/27)^{\frac{1}{2}}$, we find that $z^3 = -\frac{N}{2} + \frac{1}{2}(N^2 + 4M^3/27)^{\frac{1}{2}}$; i.e., y and z have simply been interchanged. This is to be expected, since they enter symmetrically into the various equations such as (7), (8), and (9).

There are three cube roots of (14) and three cube roots of (15). Let Y and Z be cube roots of (14) and (15) respectively which satisfy condition (9), namely, $3YZ = -M$. The other cube roots of (14) are ωY and $\omega^2 Y$, and the other cube roots of (15) are ωZ and $\omega^2 Z$, where ω and ω^2 are the imaginary cube roots of unity, namely,

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

These roots may be paired to satisfy (9) as follows:

$$3 \cdot \omega Y \cdot \omega^2 Z = -M, \qquad 3 \cdot \omega^2 Y \cdot \omega Z = -M.$$

If these pairs of values are substituted in (7), we obtain, for the roots of the reduced cubic,

$$p_1 = Y + Z, \quad p_2 = \omega Y + \omega^2 Z, \quad p_3 = \omega^2 Y + \omega Z. \qquad (16)$$

Making use of (4) and (3), we find for the roots of (1),

$$\begin{aligned} x_1 &= Y + Z - \frac{a_1}{3a_0}, \\ x_2 &= \omega Y + \omega^2 Z - \frac{a_1}{3a_0}, \qquad (17) \\ x_3 &= \omega^2 Y + \omega Z - \frac{a_1}{3a_0}. \end{aligned}$$

These expressions for the roots of a cubic are called Cardan's formulas.

VITA

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